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ON THE DEGREE OF INSTABILITY†

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The degree of instability of an equilibrium position in an autonomous dynamical system is defined as the number of eigenvalues of its linearization that lie in the right half-plane. Dissipative systems with Morse functions that do not increase along their trajectories are considered. The critical points of such functions are precisely the equilibrium positions. It will be shown that the degree of instability of a non-degenerate equilibrium position has the same parity as the index of the Morse function at that point. In particular, if the index is odd, the equilibrium is unstable. This result carries over to compact invariant manifolds of a dynamical system, provided they are non-degenerate, reducible and ergodic. An example is the problem of the stability of the steady motion of a heavy cylindrical rigid body in an unbounded volume of ideal liquid with non-zero circulation.

1. DEGREE OF INSTABILITY

LET υ BE a smooth vector field on an n-dimensional manifold M with local coordinates $x=(x_1,\ldots,x_n)$. It generates a dynamical system on M

$$x^{\cdot} = \upsilon(x) \tag{1.1}$$

We shall assume throughout this paper that $x=0$ is an equilibrium position, $\upsilon(0)=0$. In the neighbourhood of this point, then, system (1.1) may be written $x^{\cdot}=Ax+o(|x|)$, where A is the Jacobian of υ at $x=0$.

The *degree of instability* of the equilibrium $x=0$ is defined as the number of eigenvalues of A (counting multiplicities) with positive real part. This definition is a natural generalization of the definition of the degree of instability for equilibrium positions of reversible mechanical systems, proposed by Poincaré [1]. In particular, if the degree of instability is odd, then the characteristic polynomial of A has a positive real root.

The degree of instability may also be defined for a reducible compact invariant manifold N of system (1.1). Let y be local coordinates on N, and z the coordinates in the transversal direction. In these variables N is defined by the equation $z=0$, while Eqs (1.1) take the following form

$$y^{\cdot} = u(y)+f(y,z), \qquad z^{\cdot}=Az+g(y,z)$$
$$f(y,0) = 0, \qquad g=O(|z|^2) \tag{1.2}$$

where u is the restriction of the field υ to N. By our reducibility assumption, the matrix A is constant in suitable coordinates y, z. If $\dim N=1$ (i.e. N is a periodic trajectory), Eqs (1.1) are

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always reducible (the Floquet–Lyapunov theorem). The degree of instability of N is again defined a the number of eigenvalues of A with positive real part; we will denote it by $\deg N$.

It is a rather complicated problem to determine whether invariant manifolds of dimensions $\geqslant 2$ are reducible. It has not been completely solved even for a torus (see [2]).

An invariant manifold N is said to be *non-degenerate* if $|A| \neq 0$. The following lemma is easy to prove.

Lemma 1. Let N be a reducible non-degenerate invariant manifold. Then

$$\deg N \equiv \tfrac{1}{2}(1 - \operatorname{sign}|-A|) \bmod 2 \tag{1.3}$$

Indeed, since complex eigenvalues occur in pairs, $\deg N$ is equal modulo 2 to the number of positive real roots of the characteristic polynomial (counting multiplicities). Furthermore, the sign of the coefficient of λ^n in the characteristic polynomial of A is just the sign of $|-A|$. It remains to apply Descartes' theorem on the number of positive roots of a polynomial.

One corollary of this result is as follows. Let us assume that system (1.1) depends on a parameter α and has a family of reducible non-degenerate invariant manifolds N_α which depends smoothly on α. If there is an α for which N_α has odd degree of instability of N_α, then N_α will be unstable for all values of α.

Existence conditions for invariant manifolds of dimensions $\geqslant 2$ for smooth perturbations of the initial system were studied in [3]. There, however, stringent conditions were imposed on the invariant manifold. But if $\dim N \leqslant 1$, non-degeneracy guarantees continuability of the equilibrium position or periodic solution with respect to α.

Let us consider equilibrium positions in greater detail. Let γ be a smooth regular curve in the $(n+1)$-dimensional space $M \times \mathbf{R}_\alpha$ ($\mathbf{R}_\alpha$ being the real axis in α space), which is one of the equilibrium curves: if $(x_*, \alpha_*) \in \gamma$, then $x = x_*$ is an equilibrium of system (1.1) at $\alpha = \alpha_*$. Now let

$$\ldots, (x_k, \alpha_0), \quad (x_{k+1}, \alpha_0), \ldots$$

be the points at which γ cuts the "plane" $\alpha = \alpha_0$ transversally, as arranged in sequence on γ.

Proposition 1. The difference $\deg x_{k+1} - \deg x_k$ is odd for all k.

This is an analogue of Poincaré's result concerning the law of stability reversal (see [1]). Chetayev [4] derived an analogous relationship for the indices of singular points, using the well-known Poincaré–Kronecker theorem on the sum of indices. Since the index of a non-degenerate singular point equals ± 1, depending on the sign of $|A|$, Proposition 1 is a corollary of Chetayev's theorem and our Lemma 1.

2. DISSIPATIVE SYSTEMS

If a smooth function $F: M \to \mathbf{R}$ exists such that $F^{\bullet} = (\partial F/\partial x, \ v) \leqslant 0$, we shall call the system *dissipative*. The function F plays a role similar to that of the total energy. It has been shown [5] that the non-degenerate critical points of F correspond to the equilibrium positions of system (1.1).

Our main result is the following theorem.

Theorem 1. Let x_0 be a non-degenerate equilibrium position of system (1.1) which is a non-degenerate critical point of F. Then

$$\deg x_0 \equiv \operatorname{ind}_{x_0} F \pmod 2 \tag{2.1}$$

The expression on the right of this equality is the index of F at the critical point x_0.

Corollary. Let F be a Morse function. Then its critical points of odd index are unstable equilibria of system (1.1).

If F is an integral of (1.1), this statement was proved in [6] (for a simple proof, see [7]). If n is even, the condition $F^{\cdot} \leqslant 0$ may be replaced by $F^{\cdot} \geqslant 0$.

This corollary is a natural generalization of the Kelvin–Chetayev theorem according to which the equilibrium of a mechanical system with odd Poincaré degree of instability is unstable when arbitrary gyroscopic and dissipative forces are added. Indeed, let F be an energy integral of a reversible system. Its index in an equilibrium position is obviously odd, and this situation is maintained on adding the gyroscopic forces. Since, after adding the dissipative forces $F^{\cdot} \leqslant 0$, the instability of the equilibrium follows from the corollary.

Proof of Theorem 1. Put $\Phi = -F$ and let $x=0$ be a non-degenerate critical point of Φ. In its neighbourhood, $\Phi = \Phi(0) + (Bx,\ x)/2 + o(|x|^2)$. Since $\Phi^{\cdot} \geqslant 0$, the quadratic form $(x,\ BAx)$ is non-negative. Put $D=(BA+A^{\mathrm{T}}B)/2$. Consequently, the symmetric matrix D is also non-negative. Since by assumption A and B are non-singular matrices, it follows that $|C| \neq 0$. Since $(C+C^{\mathrm{T}})/2 = D$, it follows that $C = D+J$, where J is skew-symmetric.

The following algebraic lemma holds.

Lemma 2. If $D \geqslant 0$, then $|D+J| \geqslant 0$ for any skew-symmetric matrix J.

For simplicity, we will consider the case $n=3$. A non-singular matrix K exists such that $K^{\mathrm{T}}DK$ is diagonal. Since $|KK^{\mathrm{T}}| > 0$, the sign of the determinant

$$|K^{\mathrm{T}}DK + K^{\mathrm{T}}JK| = |K^{\mathrm{T}}K||D+J|$$

will be the same as that of $|D+J|$. Since $K^{\mathrm{T}}JK$ is skew-symmetric, we may assume that D has already been reduced to diagonal form. Next

$$\begin{vmatrix} \lambda & a & b \\ -a & \mu & c \\ -b & -c & \nu \end{vmatrix} = \lambda\mu\nu + \lambda c^2 + \mu b^2 + \nu a^2 \geqslant 0$$

if the numbers λ, μ and ν are non-negative, which is what was required.

Thus, by Lemma 2, $|C| > 0$. Hence $\operatorname{sign}|-A||-B| > 0$. Clearly, $\operatorname{sign}|-B| = (-1)^{\operatorname{ind} F}$. Consequently, by Lemma 1, $\deg(0) \equiv \frac{1}{2}[1-(-1)^{\operatorname{ind} F}] \bmod 2$. If F has an even index, then $\deg(0) \equiv 0 (\bmod 2)$, but if the index is odd, then $\deg(0) = 1 (\bmod 2)$. This completes the proof of the theorem.

3. SOME GENERALIZATIONS

Let N be a connected compact reducible m-dimensional invariant manifold of a dynamical system (1.1) whose restriction to N has an invariant measure with density $\rho > 0$. In the neighbourhood of N Eqs (1.1) have the form (1.2). The restriction of system (1.1) to N is defined by the equation

$$y^{\cdot} = u(y), \quad y \in N \tag{3.1}$$

Let

$$F(y,z) = F_0(y) + (z, h(y)) + (B(y)z, z)/2 + \dots$$

be a smooth function defined in the neighbourhood of N, with $F^{\cdot} \leqslant 0$. Clearly, F_0 is a smooth function on N and its derivative along trajectories of system (3.1) is non-positive.

Lemma 3. If system (3.1) is ergodic, then $F_0 = \text{const}$.

The simplest example of an ergodic system is conditionally-periodic motion on an m-dimensional torus $N = \mathbf{T}^m = \{y_1, \ldots, y_m \bmod 2\pi\}$

$$y_1^{\cdot} = \omega_1, \ldots, \quad y_m^{\cdot} = \omega_m; \quad \omega_j = \text{const} \tag{3.2}$$

In the typical case of incommensurable frequencies $\omega_1, \ldots, \omega_n$, the system is ergodic. In particular, this relates to periodic trajectories $(m = 1)$.

Proof of Lemma 3. Put $F_0^{\cdot} = \Phi \leqslant 0$. Then

$$\frac{1}{T}[F_0(y(T)) - F_0(y(0))] = \frac{1}{T}\int_0^T \Phi(y(t))dt \tag{3.3}$$

Since F_0 is continuous on the compact set N, it is bounded. Consequently, the limit of the left-hand side of (3.3) as $T \to \infty$ is zero. On the other hand, by Birkhoff's ergodic theorem [8], the limit of the right-hand side of (3.3) is

$$\frac{1}{\text{mes} N}\int_N \rho\Phi d^m y, \qquad \text{mes} N = \int_N \rho d^m y$$

Since $\rho > 0$, $\Phi \leqslant 0$ and this integral vanishes, we obtain $\Phi = 0$. Consequently, F_0 is an integral of system (3.1). By ergodicity, $F_0 = \text{const}$. This proves the lemma.

The terms that are linear in z in the expression for $F^{\cdot}$ may be written

$$(Az, h) + (z, h^{\cdot}) \tag{3.4}$$

where $h^{\cdot}$ is the derivative of the covector field h along trajectories of system (3.1). Since $F^{\cdot} \leqslant 0$, the sum (3.4) must vanish. Hence

$$(\partial h/\partial y, u) = -A^{\tau} h \tag{3.5}$$

An invariant manifold N is said to be *strongly non-degenerate* if the only solution of Eq. (3.5) is zero. Strong non-degeneracy implies ordinary non-degeneracy of N as defined in Sec. 1 (otherwise Eq. (3.5) would have a non-trivial solution $h = \text{const}$). For the invariant manifold (3.2), strong degeneracy means that A has no eigenvalues of the form $i(k_1\omega_1 + \ldots + k_m\omega_m)$, $k_j \in \mathbf{Z}$. For periodic trajectories $(m = 1)$ it is equivalent to the condition that the multipliers be different from unity.

Thus, under our assumptions the Taylor series of F around N begins with a quadratic form in z. We average this form over the invariant manifold N

$$\Phi(z) = \frac{1}{2}\int_N (B(y)z, z)\rho(y)d^m y \tag{3.6}$$

Theorem 2. Suppose that the connected compact invariant manifold N is reducible, ergodic and strongly non-degenerate. If the quadratic form (3.6) is non-degenerate, then $\deg N \equiv \operatorname{ind}\Phi \pmod 2$.

Corollary. Under the assumptions of Theorem 2, if the form Φ has odd index, then N is unstable.

This proposition may be considered a partial inversion of a theorem on the stability of invariant manifolds established in [5]. If F is an integral of system (1.1), instability was proved in [7]. The proof of Theorem 2 is exactly the same as that of Theorem 1, using the scheme suggested in [7].

4. SOME APPLICATIONS

As an example, let us consider a two-dimensional problem in hydrodynamics: a heavy cylindrical solid falling in an unbounded volume of an ideal fluid in which there is a plane-parallel irrotational flow at rest at infinity. It is assumed that the generators of the cylinder are orthogonal to the flow plane. By Thomson's theorem, the circulation Γ of the fluid about the cylinder is constant.

In a suitable frame of reference $O\xi\eta\zeta$ attached to the body (with the $O\zeta$ axis orthogonal to the flow plane), the kinetic energy of the "body plus fluid" system may be written in the form

$$(a_1u^2 + a_2\upsilon^2 + b\varphi^{\cdot 2})/2$$

where u and υ are the projections of the velocity of the point O on to the $O\xi$ and $O\eta$ axes, and Φ is the angle of rotation of the body. The coefficients a_1 and a_2 include the added masses and added moment of inertia.

The equations of motion of the body in the fluid may be put in the form of Kirchhoff's equations [9, Sec. 134a]

$$\begin{aligned} &a_1u^{\cdot} + a_2\upsilon\varphi^{\cdot} + \lambda\upsilon = -p\cos\varphi, \qquad a_2\upsilon^{\cdot} - a_1u\varphi^{\cdot} - \lambda u = -p\sin\varphi \\ &b\varphi^{\cdot\cdot} + (a_1 - a_2)u\upsilon = p(\xi\sin\varphi - \eta\cos\varphi) \end{aligned} \tag{4.1}$$

where $\lambda = \rho\Gamma$, ρ is the fluid density and p is the weight of the body minus the Archimedes force; for a two-dimensional homogeneous body ξ and η are the Cartesian coordinates of its centre of mass. We shall assume that $a_2 > a_1$ and $\lambda \neq 0$.

Steady motions of this body and their stability were studied in [10]. The positions of equilibrium of system (4.1) are defined by the equations

$$\lambda u = p\sin\varphi, \quad \lambda\upsilon = -p\cos\varphi, \quad \varphi = \varphi_* \tag{4.2}$$

where φ_* is the root of the equation

$$\begin{aligned} g(\varphi) &= \alpha\sin\varphi + \beta\cos\varphi + \sin\varphi\cos\varphi = 0 \\ \alpha &= -\xi\lambda^2/[p(a_2 - a_1)], \quad \beta = \eta\lambda^2/[p(a_2 - a_1)] \end{aligned}$$

Equations (4.1) have an integral

$$F = \frac{a_1}{2}\left(u - \frac{p}{\lambda}\sin\varphi\right)^2 + \frac{a_2}{2}\left(\upsilon + \frac{p}{\lambda}\cos\varphi\right)^2 + \frac{b}{2}\varphi^{\cdot 2} + \frac{p^2(a_2 - a_1)}{\lambda^2}G(\varphi)$$

where G is the primitive of g. If $\varphi = \varphi_*$ is a strict local minimum of G, the steady motion (4.2) is stable by Lyapunov's theorem. But if G has a local maximum there, the equilibrium (4.2) will be non-degenerate and the index of F at the point will be unity. Consequently, by Theorem 1, the equilibrium (4.2) will be unstable. Note that F is not the total mechanical energy of the system.

The non-degenerate minima and maxima of G alternate. Hence it follows that values of φ corresponding to stable and unstable motions also alternate. This phenomenon, already pointed out in [10], is a special case of the stability reversal law established by Proposition 1.

On the right-hand side of the third equation of system (4.1) one can add a dissipative term $-\mu\varphi^{\cdot}$, where μ is a positive coefficient, which may depend on the position of the solid body. This does not affect the steady solutions (4.2). Since $F^{\cdot} \leqslant 0$, the above stability conclusions remain valid.

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REFERENCES

1. CHETAYEV N. G., *Stability of Motion.* Gostekhizdat, Moscow, 1955.
2. BOGOLYUBOV N. N., MITROPOL'SKII Yu. A. and SAMOILENKO A. M., *The Method of Accelerated Convergence in Non-linear Mechanics.* Naukova Dumka, Kiev, 1969.
3. TRESHCHEV D. V., On the conservation of invariant manifolds of Hamiltonian systems under perturbation. *Mat. Zametki* **50**, 4, 123–131, 1991.
4. CHETAYEV N. G., Kronecker characteristics. Uchen. Zap. Kazan. Univ., Vol. 98, Book 9, *Mathematics, Fasc.* **3**, 1–41, 1938.
5. KARAPETYAN A. V., The Routh theorem and its extensions. In *Colloq. Math. Soc. Janos Bolyai, 53: Qualitative Theory of Differential Equations*, pp. 271–290. Szeged (1988). North-Holland, Amsterdam, 1990.
6. RUBANOVSKII V. N., On the bifurcation and stability of steady motions. *Teor. Prilozh. Mekh.* **5**, 1, 67–79, 1974.
7. KOZLOV V. V., Linear systems with a quadratic integral. *Prikl. Mat. Mekh.* **56**, 6, 900–906, 1992.
8. NEMYTSKII V. V. and STEPANOV V. V., *The Qualitative Theory of Differential Equations.* Gostekhizdat, Moscow, 1949.
9. LAMB H., *Hydrodynamics,* 6th Edn. Dover, New York, 1945.
10. KOZLOV V. V., On falling of a heavy cylindrical solid body in fluid with non-zero circulation. *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela* 4, 113–117, 1993.

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